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DUALITY IN THE ANALYSIS OF SHELLS BY THE FINITE ELEMENT METHOD

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Abstract—The static–geometric duality is a property particularly useful for application of the finite element method to the analysis of doubly curved shells. After establishing the boundary conditions for Kirchhoff–Love shells and finding the duality by the energy variational principle, this paper points out that, to every conforming displacement element, one can let correspond an equilibrium element, and vice-versa. This is a generalization of the slab analogy between the bending of plates and the stretching of membranes discovered some years ago by Fraeijs de Veubeke and Zienkiewicz.

NOTATION

α_1, α_2	orthogonal curvilinear coordinates
A_1, A_2	Lamé parameters
R_1, R_2	principal radiuses of curvature of the middle surface
ρ_1, ρ_2	radiuses of geodesic curvature
u, v, w	displacement components
U, V, W	dual stress functions
$\varepsilon_1, \varepsilon_2, \omega_1, \omega_2$	components of membrane strain tensor
$\kappa_1, \kappa_2, \tau_1, \tau_2$	components of curvature tensor
φ_1, φ_2	rotation components
N_1, N_2, N_{12}, N_{21}	membrane force resultants
M_1, M_2, M_{12}, M_{21}	bending and torsion moment resultants
Q_1, Q_2	transverse shear resultants
τ	stress tensor in matrix form
ε	strain tensor in matrix form
$[N]$	force matrix
$[M]$	moment matrix
$[\chi]$	curvature matrix
$[H]$	Hooke's matrix
δW	strain energy variation
$\delta \phi$	complementary energy variation
E	Young's modulus
ν	Poisson's ratio
h	shell thickness

1. INTRODUCTION

THE finite element method is one of the most powerful numerical methods discovered recently [1] for the analysis of complicated structures. It consists of dividing the structure in a certain number of finite elements with simple geometric shape and structural function. In each element, suitable assumptions are made on the displacement field or the stress field, or both at the same time. In this manner displacement models, equilibrium models or hybrid models are obtained.

Among the displacement and equilibrium models there are two sets of dual elements distinguished from others by their special properties. Thus, a displacement model is called conforming if its displacement field satisfies *a priori* the internal compatibility conditions and secures the continuity of this field on the boundary. Likewise, an equilibrium model is faultless if the internal equilibrium conditions are fulfilled *a priori* by the assumed stress field and continuous stress transmission is warranted at the edge. As shown in Ref. [2], the first set of models yields an upper bound and the second, a lower bound of the direct static influence coefficients. This property represents an important tool for the determination of the quality and the numerical evaluation of the element.

Although the construction of the conforming displacement models is not easy, it presents fewer difficulties than that of the equilibrium models. A successful attempt to get new possibilities of development of the last models has been made by Fraeijs de Veubeke and Zienkiewicz [3] and afterwards by Elias Ziad [4]. They followed up on Southwell's analogy and proved that each conforming displacement element of membrane stretching corresponds to an equilibrium element of plate bending, and vice versa.

In this paper, based on the static–geometric analogy of Lure [5] and Goldenveizer [6], the duality is shown to be a general property of shell theory, of which plates and membranes form particular cases.

2. FORMULATION OF THE GENERAL THEORY OF SHELLS

The general theory of the Kirchhoff–Love shells was first proposed in 1874 by Aron [8]. Generalizations and corrections were formulated by Love [9], Goldenveizer [5], Lure [6] and Novozhilov [7].

In this section, the fundamental results from the theory of thin shells are collected as presented by Novozhilov in his above mentioned book [7].

2.1 *Strain relation*

Let α_1, α_2 be an orthogonal curvilinear coordinate system which coincides with the principal lines of curvature. The following symbols are defined

A_1, A_2 corresponding Lamé parameters (first fundamental quadratic form of the surface)
R_1, R_2 principal radiuses of curvature of the middle surface
ρ_1, ρ_2 radiuses of geodesic curvature on the middle surface

$$\frac{1}{\rho_1} = \frac{1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_2}; \qquad \frac{1}{\rho_2} = \frac{1}{A_2A_1}\frac{\partial A_1}{\partial \alpha_1}$$

u, v displacement components of a point on the middle surface
w vertical deflection of a point on the middle surface.

The components of strain of the middle surface are obtained as

$$\left.\begin{aligned}
\varepsilon_1 &= \frac{1\partial u}{A_1\partial\alpha_1}+\frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}v+\frac{w}{R_1} = \frac{\partial u}{A_1\partial\alpha_1}+\frac{v}{\rho_1}+\frac{w}{R_1}\\
\varepsilon_2 &= \frac{1\partial v}{A_2\partial\alpha_2}+\frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}u+\frac{w}{R_2} = \frac{\partial v}{A_2\partial\alpha_2}+\frac{u}{\rho_2}+\frac{w}{R_2}\\
\omega_1 &= \frac{1\partial y}{A_1\partial\alpha_1}-\frac{\partial A_1}{A_1A_2\partial\alpha_2}u = \frac{1\partial v}{A_1\partial\alpha_1}-\frac{u}{\rho_1}\\
\omega_2 &= \frac{1\partial u}{A_2\partial\alpha_2}-\frac{\partial A_2}{A_1A_2\partial\alpha_1}v = \frac{1\partial u}{A_2\partial\alpha_2}-\frac{v}{\rho_2}
\end{aligned}\right\} \tag{1}$$

$$\left.\begin{aligned}
\kappa_1 &= \frac{1\partial}{A_1\partial\alpha_1}\frac{1\partial w}{A_1\partial\alpha_1}-\frac{u}{R_1}-\frac{1}{\rho_1}\left(\frac{1\partial w}{A_2\partial\alpha_2}-\frac{v}{R_2}\right)\\
\kappa_2 &= -\frac{1\partial}{A_2\partial\alpha_2}\left(\frac{1\partial w}{A_2\partial\alpha_2}-\frac{v}{R_2}\right)-\frac{1}{\rho_2}\left(\frac{1\partial w}{A_1\partial\alpha_1}-\frac{u}{R_1}\right)\\
\tau_1 &= \frac{1\partial}{A_1\partial\alpha_1}\varphi_2-\frac{\varphi_1}{\rho_1}; \qquad \varphi_1 = -\frac{1\partial w}{A_1\partial\alpha}+\frac{u}{R_1}\\
\tau_2 &= +\frac{1\partial\varphi_1}{A_2\partial\alpha_2}-\frac{\varphi_2}{\rho_2}; \qquad \varphi_2 = -\frac{1\partial w}{A_2\partial\alpha_2}+\frac{v}{R_2}
\end{aligned}\right\} \tag{2}$$

We will also employ the following notations:

$$\left.\begin{aligned}
\theta &= \frac{1}{2A_1A_2}\left[\frac{\partial A_1u}{\partial\alpha_2}-\frac{\partial A_2v}{\partial\alpha_1}\right] = \frac{\omega_2-\omega_1}{2}\\
\omega &= \frac{\omega_1+\omega_2}{2} = \frac{1}{2}\left[\frac{A_2\partial}{A_1\partial\alpha_1}\left(\frac{v}{A_2}\right)+\frac{A_1\partial}{A_2\partial\alpha_2}\left(\frac{u}{A_1}\right)\right].
\end{aligned}\right\} \tag{3}$$

2.2 *Compatibility equations*

The three well known Gauss–Godazzi equations must be satisfied by the deformed surface, so that the train components are related by the three compatibility equations:

$$\left.\begin{aligned}
&\frac{\partial}{\partial\alpha_1}(A_2\kappa_2^*)+\frac{\partial A_1}{\partial\alpha_2}\tau_2^*-\frac{\partial}{\partial\alpha_2}(A_1\tau_1^*)-\frac{\partial A_2}{\partial\alpha_1}\kappa_1^*\\
&\qquad -\frac{1}{R_1}\left[\frac{\partial}{\partial\alpha_1}(A_2\varepsilon_2^*)+\frac{\partial A_1}{\partial\alpha_2}\omega_2^*-\frac{\partial}{\partial\alpha_2}(A_1\omega_1^*)-\frac{\partial A_2}{\partial\alpha_1}\varepsilon_1^*\right] = 0\\
&\frac{\partial}{\partial\alpha_2}(A_1\kappa_1^*)+\frac{\partial A_2}{\partial\alpha_1}\tau_1^*-\frac{\partial}{\partial\alpha_1}(A_2\tau_2^*)-\frac{\partial A_1}{\partial\alpha_1}\kappa_2^*\\
&\qquad -\frac{1}{R_2}\left[\frac{\partial}{\partial\alpha_2}(A_1\varepsilon_1^*)+\frac{\partial A_2}{\partial\alpha_1}\omega_1^*-\frac{\partial}{\partial\alpha_1}(A_2\omega_2^*)-\frac{\partial A_1}{\partial\alpha_2}\varepsilon_2^*\right] = 0\\
&A_1A_2\left(\frac{\kappa_2^*}{R_1}+\frac{\kappa_1^*}{R_2}\right)-\frac{\partial}{\partial\alpha_2}\left\{\frac{1}{A_2}\left[-\frac{\partial}{\partial\alpha_1}(A_2\omega_2^*)+\frac{\partial A_1}{\partial\alpha_2}\varepsilon_2-\frac{\partial}{\partial\alpha_2}(A_1\varepsilon_1^*)+\frac{\partial A_2}{\partial\alpha_1}\omega_1^*\right]\right\}\\
&\qquad +\frac{\partial}{\partial\alpha_2}\left\{\frac{1}{A_1}\left[\frac{\partial}{\partial\alpha_1}(A_2\varepsilon_2^*)+\frac{\partial A_1}{\partial\alpha_2}\omega_2^*-\frac{\partial}{\partial\alpha_2}(A_1\omega_1^*)-\frac{\partial A_2}{\partial\alpha_1}\varepsilon_1^*\right]\right\} = 0,
\end{aligned}\right\} \tag{4}$$

where

$$
\left.
\begin{aligned}
&\varepsilon_1 = \varepsilon_1^*; \qquad \varepsilon_2 = \varepsilon_2^*; \qquad \kappa_1 = \kappa_1^*; \qquad \kappa_2 = \kappa_2^* \\
&\omega_1^* = \frac{\omega_1+\omega_2}{2}; \qquad \omega_2^* = \frac{\omega_1+\omega_2}{2} \\
&\tau_1^* = \tau_1 + \frac{\omega_2-\omega_1}{2R_1}; \qquad \tau_2^* = -\tau_2 + \frac{\omega_2-\omega_1}{2R_2}.
\end{aligned}
\right\} \tag{5}
$$

2.3 *Equilibrium equations*

The equivalent forces and moments acting on the sides of the shell are defined in Fig. 1.

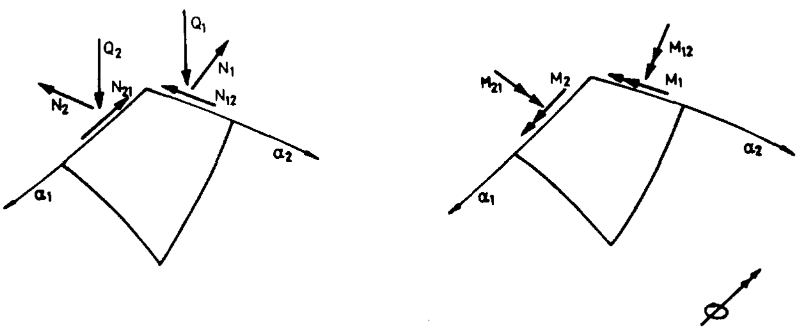

FIG. 1. Definitions of the force and moment resultants of the shell.

Let q_1, q_2, q_n be the three components of the external forces acting on a unit area of the middle surface. The following exact equilibrium equations are obtained by projecting all forces on the three directions of the coordinate axes

$$
\left.
\begin{aligned}
&\frac{1}{A_1A_2}\left[\frac{\partial A_2N_1}{\partial\alpha_1}+\frac{\partial A_1N_{21}}{\partial\alpha_2}+\frac{\partial A_1}{\partial\alpha_2}N_{12}-\frac{\partial A_2}{\partial\alpha_1}N_2\right]+\frac{Q_1}{R_1}+q_1 = 0 \\
&\frac{1}{A_1A_2}\left[\frac{\partial A_2N_{12}}{\partial\alpha_1}+\frac{\partial A_1N_2}{\partial\alpha_2}+\frac{\partial A_2}{\partial\alpha_1}N_{21}-\frac{\partial A_1}{\partial\alpha_2}N_1\right]+\frac{Q_2}{R_2}+q_2 = 0 \\
&\frac{1}{A_1A_2}\left[\frac{\partial A_2N_1}{\partial\alpha_1}+\frac{\partial A_1N_2}{\partial\alpha_2}\right]-\frac{N_1}{R_1}-\frac{N_2}{R_2}+q_n = 0
\end{aligned}
\right\}, \tag{6}
$$

$$
\left.
\begin{aligned}
&\frac{1}{A_1A_2}\left[\frac{\partial A_2M_1}{\partial\alpha_2}+\frac{\partial A_1M_{21}}{\partial\alpha_2}+\frac{\partial A_1}{\partial\alpha_2}M_{12}-\frac{\partial A_2}{\partial\alpha_1}M_2\right]-Q_1 = 0 \\
&\frac{1}{A_1A_2}\left[\frac{\partial A_2M_{12}}{\partial\alpha_1}+\frac{\partial A_1M_2}{\partial\alpha_2}+\frac{\partial A_2}{\partial\alpha_1}M_{21}-\frac{\partial A_1}{\partial\alpha_2}M_1\right]-Q_2 = 0
\end{aligned}
\right\}. \tag{7}
$$

2.4 *Strain energy*

For a thin shell, it is possible to express the strain energy in terms of the components of strain and stress of the middle surface:

$$\delta W = \iint \Big[N_1 \delta\varepsilon_1 + N_2 \delta\varepsilon_2 + N_{12}\delta\omega_1 + N_{21}\delta\omega_2 + M_1 \delta\kappa_1 + M_2 \delta\kappa_2 + M_{12}\delta\tau_1 + M_{21}\delta\tau_2 \Big] \times A_1 A_2 \, d\alpha_1 \, d\alpha_2 . \tag{8}$$

Putting

$$S = N_{12} - \frac{M_{21}}{R_2} = N_{21} - \frac{M_{12}}{R_1}$$

$$H = \tfrac{1}{2}(M_{12} + M_{21})$$

$$\omega = \omega_1 + \omega_2$$

$$\tau = \tau_1 + \frac{\omega_2}{R_1} = \tau_2 + \frac{\omega_1}{R_2},$$

where

$$\sigma^T = |N_1 N_2 S M_1 M_2 2H|$$

$$\varepsilon^T = |\varepsilon_1 \varepsilon_2 \omega \kappa_1 \kappa_2 \tau|,$$

the variation of the strain energy is finally obtained as

$$\delta W = \iint [\sigma^T \delta\varepsilon] A_1 A_2 \, d\alpha_1 \, d\alpha_2 \tag{9}$$

2.5 *Strain–stress relations*

Hooke's law for a thin shell may be written in matrix form as

$$\sigma = [H]\varepsilon, \tag{10}$$

where

$$[H] = \begin{vmatrix} M & 0 \\ 0 & N \end{vmatrix}$$

and

$$M = \frac{Eh}{1-\nu^2} \begin{vmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{vmatrix}$$

$$N = \frac{Eh^3}{12(1-\nu^2)} \begin{vmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu) \end{vmatrix}$$

Then, the expression of strain energy is

$$W=\frac{1}{2}\iint[\varepsilon^T[H]\varepsilon]A_1A_2\,d\alpha_1\,d\alpha_2=\frac{1}{2}\iint[\sigma^T[H]^{-1}\sigma]A_1A_2\,d\alpha_1\,d\alpha_2,$$

or, explicitly:

$$\begin{aligned}W=&\frac{Eh}{2(1-\nu^2)}\iint\left[(\varepsilon_1+\varepsilon_2)^2-2(1-\nu)\left(\varepsilon_1\varepsilon_2-\frac{\omega^2}{4}\right)\right]A_1A_2\,d\alpha_1\,d\alpha_2\\&+\frac{Eh^3}{24(1-\nu^2)}\iint[(\kappa_1+\kappa_2)^2-2(1-\nu)(\kappa_1\kappa_2-\tau^2)]A_1A_2\,d\alpha_1\,d\alpha_2,\end{aligned}\tag{11}$$

where h is the thickness of the shell.

3. BOUNDARY CONDITIONS OF KIRCHHOFF–LOVE SHELLS BY THE VARIATIONAL METHOD

The variation with respect to the displacements of the strain energy of the shells defined by (8) may be written in the following form

$$\begin{aligned}\delta W=&\iint[N_1\delta\varepsilon_1+N_2\delta\varepsilon_2+N_{12}\delta\omega_1+N_{21}\delta\omega_2+M_1\delta\kappa_1+M_2\delta\kappa_2+M_{12}\delta\tau_1+M_{21}\delta\tau_2]\\&\times A_1A_2\,d\alpha_1\,d\alpha_2+\iint\left[\left(N_{12}-N_{21}+\frac{M_{12}}{R_1}-\frac{M_{21}}{R_2}\right)\delta\left(\frac{\omega_2-\omega_1}{2}\right)\right]A_1A_2\,d\alpha_1\,d\alpha_2,\end{aligned}$$

where the newly introduced bracket

$$N_{12}-N_{21}+\frac{M_{12}}{R_1}-\frac{M_{21}}{R_2}$$

is shown to vanish as a consequence of the definitions

$$N_{12}=\int_{-h/2}^{h/2}\sigma_{12}\left(1+\frac{z}{R_2}\right)dz\qquad N_{21}=\int_{-h/2}^{h/2}\sigma_{21}\left(1+\frac{z}{R_1}\right)dz$$

$$M_{12}=\int_{-h/2}^{h/2}z\left(1+\frac{z}{R_2}\right)\sigma_{12}\,dz\qquad M_{21}=\int_{-h/2}^{h/2}z\left(1+\frac{z}{R_1}\right)\sigma_{21}\,dz$$

of N_{12}, N_{21}, M_{12} and M_{21}. Indeed,

$$N_{12}-N_{21}+\frac{M_{12}}{R_1}-\frac{M_{21}}{R_2}=\int_{-h/2}^{h/2}\left(1+\frac{z}{R_1}\right)\left(1+\frac{z}{R_2}\right)(\sigma_{12}-\sigma_{21})\,dz=0,$$

because of the symmetry of the stress tensor. Now, the strain energy variation is arranged in the form:

$$\begin{aligned}\delta W=\iint\Big[&N_1\delta\varepsilon_1+N_2\delta\varepsilon_2+N_{12}\delta\left(\frac{\omega_1+\omega_2}{2}\right)+N_{21}\delta\left(\frac{\omega_2+\omega_1}{2}\right)+M_1\delta\kappa_1+M_2\delta\kappa_2\\&+M_{12}\delta\left(\tau_1+\frac{\omega_2-\omega_1}{2R_1}\right)+M_{21}\delta\left(\tau_2-\frac{\omega_2+\omega_1}{2R_1}\right)\Big]A_1A_2\,d\alpha_1\,d\alpha_2.\end{aligned}$$

Putting

$$\left.\begin{array}{llll} \varepsilon_1^* = \varepsilon_2; & \varepsilon_2^* = \varepsilon_2; & \omega_1^* = \omega_2^* = \dfrac{\omega_2+\omega_1}{2} & \\ \kappa_1^* = \kappa_1; & \kappa_2^* = \kappa_2; & \tau_1^* = \tau_1 + \dfrac{\omega_2+\omega_1}{2R_1}; & \tau_2^* = \tau_2 - \dfrac{\omega_2-\omega_1}{2R_2} \end{array}\right\}, \tag{12}$$

the strain energy becomes

$$\delta W = \iint [N_1\delta\varepsilon_1^* + N_2\delta\varepsilon_2^* + N_{12}\delta\omega_1^* + N_{21}\delta\omega_2^* + M_1\delta\kappa_1^* + M_2\delta\kappa_2^* + M_{12}\delta\tau_1^* + M_{21}\delta\tau_2^*] \times A_1A_2\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2. \tag{13}$$

The complementary energy variation may be written immediately by permutation of variables as

$$\delta\Phi = \iint [\varepsilon_1^*\delta N_1 + \varepsilon_2^*\delta N_2 + \omega_1^*\delta N_{12} + \omega_2^*\delta N_{21} + \kappa_1^*\delta M_1 + \kappa_2^*\delta M_2 + \tau_1^*\delta M_{12} + \tau_2^*\delta M_{21}] \times A_1A_2\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2. \tag{14}$$

For convenience, the strain energy is decomposed in different terms:

$$\delta W = \delta W_1 + \delta W_2 + \delta W_3 + \delta W_4 + \delta W_5,$$

where

$$\delta W_1 = \iint [N_1\delta\varepsilon_1 + N_2\delta\varepsilon_2 + N_{12}\delta\omega_1 + N_{21}\delta\omega_2]A_1A_2\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2$$

$$\delta W_2 + \delta W_3 + \delta W_4 + \delta W_5 = \iint [M_1\delta\kappa_1 + M_2\delta\kappa_2 + M_{12}\delta\tau_1 + M_{21}\delta\tau_2]A_1A_2\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2.$$

The quantities δW_2, δW_3, δW_4, δW_5 will be defined later on.

Substituting now the membrane strain components by their values from (1), one obtains

$$\begin{aligned}\delta W_1 = \iint\Big\{ & \frac{\partial}{\partial\alpha_1}(N_1A_2\delta u) - \frac{\partial}{\partial\alpha_1}(N_1A_2)\delta u + N_2\frac{\partial A_1}{\partial\alpha_1}\delta u + N_1\frac{\partial A_2}{\partial\alpha_2}\delta v - \frac{\partial}{\partial\alpha_2}(N_2A_1)\delta v \\ & + \frac{N_2A_1A_2}{R_2}\delta w + N_1\frac{A_1A_2}{R_1}\delta w + \frac{\partial}{\partial\alpha_2}(N_2A_1\delta v) - \frac{\partial}{\partial\alpha_1}(A_2N_{12})\delta v - N_{12}\frac{\partial A_1}{\partial\alpha_2}\delta u \\ & + \frac{\partial(N_{12}A_2\delta v)}{\partial\alpha_1} + \frac{\partial}{\partial\alpha_2}(N_{21}A_1\delta w) - \frac{\partial}{\partial\alpha_2}(A_1N_{12})\delta u - N_{21}\frac{\partial A_2}{\partial\alpha_2}\delta u\Big\}\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2.\end{aligned}$$

Taking into consideration the Gauss–Ostrogradski formula

$$\iint\left(\frac{\partial A_2f_2}{\partial\alpha_1} + \frac{\partial A_1f_1}{\partial\alpha_2}\right)\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2 = \oint(\mathbf{n}\,.\,\mathbf{f})\,\mathrm{d}s,$$

the expression of δW_1 can be transformed by partial integration as follows

$$\delta W_1 = \oint \mathbf{n}[N]\delta\mathbf{D} - \iint \left\{ \left[\frac{\partial}{\partial\alpha_1}(N_1 A_2) + \frac{\partial}{\partial\alpha_2}(A_1 N_{21}) - N_2 \frac{\partial A_1}{\partial\alpha} + N_{12}\frac{\partial A_1}{\partial\alpha_2} \right]\delta u \right.$$
$$+ \left[\frac{\partial}{\partial\alpha_2}(N_2 A_1) + \frac{\partial}{\partial\alpha_1}(A_2 N_{12}) - N_1 \frac{\partial A_2}{\partial\alpha_2} + N_{21}\frac{\partial A_2}{\partial\alpha_1} \right]\delta v \tag{15}$$
$$\left. + \left[\frac{N_2 A_1 A_2}{R_2} + \frac{N_1 A_1 A_2}{R_1} \right]\delta w \right\} \mathrm{d}\alpha_1\, \mathrm{d}\alpha_2,$$

where the matrix $[N]$ and the displacement vector $\mathbf{D}$ are defined by

$$[N] = \begin{vmatrix} N_1 & N_{12} \\ N_{21} & N_2 \end{vmatrix}$$
$$\mathbf{D}^T = |uv|.$$

Introducing the explicit expression (2) of the components of the curvature tensor in $\delta W_2 + \delta W_3 + \delta W_4 + \delta W_5$ gives

$$\left.\begin{aligned}
\delta W_2 = {} & \iint \left\{ M_1 \left[-A_2 \frac{\partial}{\partial\alpha_1}\left(\frac{1\partial}{A_1\partial\alpha_1}\delta w \right) \right] + M_2 \left[-A_1 \frac{\partial}{\partial\alpha_2}\left(\frac{1\partial}{A_2\partial\alpha_2}\delta w \right) \right] \right. \\
& \left. + M_{12}\left[A_2 \frac{\partial}{\partial\alpha_1}\left(-\frac{1\partial}{A_2\partial\alpha_2}\delta w \right) \right] + M_{21}\left[A_1 \frac{\partial}{\partial\alpha_2}\left(-\frac{1\partial}{A_1\partial\alpha_1}\delta w \right) \right] \right\} \mathrm{d}\alpha_1\,\mathrm{d}\alpha_2 \\
\delta W_3 = {} & \iint \left\{ M_1 \left[-A_2 \frac{\partial}{\partial\alpha_1}\left(-\frac{\delta u}{R_1} \right) \right] + M_2 \left[-A_1 \frac{\partial}{\partial\alpha_2}\left(-\frac{\delta v}{R_2} \right) \right] + M_{12}\left[A_2 \frac{\partial}{\partial\alpha_2}\left(\frac{\delta u}{R_2} \right) \right] \right. \\
& \left. + M_{21}\left[A_1 \frac{\partial}{\partial\alpha_2}\left(\frac{\delta u}{R_1} \right) \right] \right\} \mathrm{d}\alpha_1\,\mathrm{d}\alpha_2
\end{aligned}\right\} \tag{16}$$

$$\left.\begin{aligned}
\delta W_4 = {} & \iint \left\{ M_1 \left[-\frac{\partial A_1}{\partial\alpha_2}\left(\frac{1\partial}{A_2\partial\alpha_2}\delta w \right) \right] + M_2 \left[-\frac{\partial A_2}{\partial\alpha_1}\left(\frac{1\partial}{A_1\partial\alpha_1}\delta w \right) \right] \right. \\
& \left. \times M_{12}\left[-\frac{\partial A_1}{\partial\alpha_2}\left(-\frac{1\partial}{A_1\partial\alpha_1}\delta w \right) \right] + M_{21}\left[-\frac{\partial A_2}{\partial\alpha_1}\left(-\frac{1\partial}{A_2\partial\alpha_2}\delta w \right) \right] \right\} \mathrm{d}\alpha_1\,\mathrm{d}\alpha_2 \\
\delta W_5 = {} & \iint \left\{ M_1 \left[-\frac{\partial A_1}{\partial\alpha_2}\left(-\frac{\delta v}{R_2} \right) \right] + M_2 \left[-\frac{\partial A_2}{\partial\alpha_1}\left(-\frac{\delta u}{R_1} \right) \right] + M_{12}\left[-\frac{\partial A_1}{\partial\alpha_2}\left(\frac{\delta u}{R_1} \right) \right] \right. \\
& \left. + M_{21}\left[-\frac{\partial A_1}{\partial\alpha_1}\left(\frac{\delta u}{R_2} \right) \right] \right\} \mathrm{d}\alpha_1\,\mathrm{d}\alpha_2 .
\end{aligned}\right\} \tag{17}$$

Integrating by parts the second term, one finds,

$$\delta W_2 = -\oint \mathbf{n}[M]\delta\, \mathbf{grad}\, w\, \mathrm{d}s + \iint \mathbf{P}_1\, \mathbf{grad}\, w\, \mathrm{d}\alpha_1\,\mathrm{d}\alpha_2, \tag{18}$$

where

$$[M] = \begin{vmatrix} M_1 & M_{12} \\ M_{21} & M_2 \end{vmatrix},$$

and the vector $\mathbf{P}_1$ is defined as

$$A_1A_2\mathbf{P}_1 = \left[\frac{\partial}{\partial\alpha_1}(M_1A_2)+\frac{\partial}{\partial\alpha_2}(M_2A_1)\right]\mathbf{e}_1+\left[\frac{\partial}{\partial\alpha_1}(M_{12}A_2)+\frac{\partial}{\partial\alpha_2}(M_2A_1)\right]\mathbf{e}_2$$
$$= \left[\frac{\partial}{\partial\alpha_1}(M_1A_2)+\frac{\partial}{\partial\alpha_2}(M_{21}A_1)\right]\frac{\mathbf{g}_1}{A_1}+\left[\frac{\partial}{\partial\alpha_1}(M_{12}A_2)+\frac{\partial}{\partial\alpha_2}(M_2A_1)\right]\frac{\mathbf{g}_2}{A_2}, \tag{19}$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ are the unit vectors along the coordinate lines q_1, α_2 and

$$\mathbf{g}_1 = \frac{\partial\mathbf{r}}{\partial\alpha_1}, \qquad \mathbf{g}_2 = \frac{\partial\mathbf{r}}{\partial\alpha_2}$$

are the partial derivatives of the position vector $\mathbf{r}$ of a point of the middle surface.

Recalling now the formula of variational analysis:

$$\mathbf{v}\,\mathbf{grad}\,\Psi = \operatorname{div}(\Psi\mathbf{v}) - \Psi\operatorname{div}\mathbf{v}$$

with

$$\operatorname{div}\mathbf{v} = \frac{1}{A_1A_2}\frac{\partial}{\partial\alpha_1}(A_1A_2v^1)+\frac{1}{A_1A_2}\frac{\partial}{\partial\alpha_2}(A_1A_2v^2),$$

the last term of the right member of equation (18) may be transformed as follows:

$$\iint \mathbf{P}_1\,\mathbf{grad}\,\delta wA_1A_2\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2 = \oint(\mathbf{nP}_1\delta w)\,\mathrm{d}s-\iint \operatorname{div}\mathbf{P}_1\delta wA_1A_2\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2.$$

Then

$$\delta W_2 = \oint(-\mathbf{n}[M]\delta\,\mathbf{grad}\,w+\mathbf{nP}_1\delta w)\,\mathrm{d}s-\iint\operatorname{div}\mathbf{P}_1\delta wA_1A_2\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2. \tag{20}$$

Applying again the Gauss–Ostrograski formula mentioned above, δW_3 is reduced to

$$\delta W_3 = \oint\mathbf{n}[M][\chi]\delta\mathbf{D}\,\mathrm{d}s-\int\mathbf{P}_1[\chi]\delta\mathbf{D}\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2A_1A_2 \tag{21}$$

where the curvature matrix is

$$[\chi] = \begin{vmatrix} \frac{1}{R_1} & \\ & \frac{1}{R_2} \end{vmatrix}.$$

In the same way,

$$\delta W_4 = \oint[\mathbf{n}\,.\,\mathbf{P}_2\delta W)]\,\mathrm{d}s-\int(\operatorname{div}\mathbf{P}_2\delta w)A_1A_2\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2 \tag{22}$$

where

$$A_1A_2\mathbf{P}_2 = \left[\left(M_{12}\frac{\partial A_1}{\partial\alpha_1}-M_2\frac{\partial A_2}{\partial\alpha_1}\right)\mathbf{e}_1+\left(M_{21}\frac{\partial A_2}{\partial\alpha_1}-M_1\frac{\partial A_1}{\partial\alpha_2}\right)\mathbf{e}_2\right]$$

As to the last term δW_5, it may be expressed in terms of $\mathbf{P}_2$ and $\mathbf{D}$. One has

$$\delta W_5 = \int\int\left\{M_1 \frac{\partial A_1}{\partial \alpha_2}\frac{\delta v}{R_2} + M_2 \frac{\partial A_2}{\partial \alpha_1}\frac{\delta u}{R_1} - M_{12}\frac{\partial A_1}{\partial \alpha_2}\frac{\delta u}{R_1} - M_{21}\frac{\partial A_2}{\partial \alpha_1}\frac{\delta v}{R_2}\right\} \mathrm{d}\alpha_1\,\mathrm{d}\alpha_2$$

Or:

$$\delta W_5 = -\int\int \mathbf{P}_2[\chi]\delta\mathbf{D} A_1 A_2\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2. \tag{23}$$

In short, the strain energy variation δW may be divided in two parts: In the interior

$$\begin{aligned}\delta W_S = &-\int\int \mathbf{Q}[\chi]\delta\mathbf{D} A_1 A_2\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2 - \int\int \operatorname{div}\mathbf{Q}\delta w A_1 A_2\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\\ &-\int\int\left\{\left[\frac{\partial}{\partial\alpha_1}(N_1A_2) + \frac{\partial}{\partial\alpha_2}(A_1N_{21}) - N_1\frac{\partial A_2}{\partial\alpha_2} + N_{12}\frac{\partial A_1}{\partial\alpha_2}\right]\delta u\right.\\ &+\left[\frac{\partial}{\partial\alpha_2}(N_2A_1) + \frac{\partial}{\partial\alpha_1}(A_2N_{12})\right.\\ &\left.\left. - N_1\frac{\partial A_2}{\partial\alpha_1} + N_{21}\frac{\partial A_2}{\partial\alpha_1}\right]\delta v + \left[\frac{N_2A_1A_2}{R_2} + \frac{N_1A_1A_2}{R_1}\right]\delta w\right\}\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\end{aligned} \tag{24}$$

where

$$\mathbf{Q} = \mathbf{P}_1 + \mathbf{P}_2.$$

At the boundary

$$\delta W_s = \oint \mathbf{n}\{[N] + [M][\chi]\}\delta\mathbf{D}\,\mathrm{d}s + \oint\{-\mathbf{n}[M]\delta\,\mathbf{grad}\,w + \mathbf{nQ}\}\,\mathrm{d}s \tag{25}$$

On the other hand, the variation of the potential energy of external forces is given by

$$\delta P = -\int\int[q_n^*\delta w + \mathbf{q}^*\delta\mathbf{D}]\,\mathrm{d}S - \oint \mathbf{n}[N^*]\delta\mathbf{D}\,\mathrm{d}s - \oint M_n^*\delta\frac{\partial w}{\partial n}\,\mathrm{d}s - \oint Q_n^*\delta w\,\mathrm{d}s - Z_i w_i \tag{26}$$

where

$\mathbf{q}^* = |q_1^* q_2^*|$ is the loading vector on the middle surface,

q_n^* = the vertical component of the surface loading.

Obviously,

$$q_1^* = A_1A_2q_1$$
$$q_2^* = A_2A_1q_2$$
$$q_n^* = A_1A_2q_n$$

Also,
N_n^* = external normal force
M_n^* = external moment
Q_n^* = external transverse force applied on the boundary and
Z_i = external concentrated transverse force.
It should also be noted that the surface element is defined as

$$dS = A_1 A_2 \, d\alpha_1 \, d\alpha_2.$$

The principle of minimum potential energy requires therefore, that

$$\delta(W+P) = 0.$$

For $\delta\mathbf{D}$ arbitrary

$$\left.\begin{aligned} \frac{\partial}{\partial\alpha_1}(N_1A_2)+\frac{\partial}{\partial\alpha_2}(A_1N_{21})-N_2\frac{\partial A_1}{\partial\alpha_1}+M_2\frac{\partial A_1}{\partial\alpha_2}+\frac{Q_1}{R_1}+q_1^* &= 0 \\ \frac{\partial}{\partial\alpha_2}(N_2A_1)+\frac{\partial}{\partial\alpha_1}(A_2N_{12})-N_1\frac{\partial A_2}{\partial\alpha_2}+M_1\frac{\partial A_2}{\partial\alpha_1}+\frac{Q_2}{R_2}+q_2^* &= 0 \end{aligned}\right\}. \tag{27}$$

For δw arbitrary

$$\frac{\partial}{\partial\alpha_1}(A_2Q_1)+\frac{\partial}{\partial\alpha_2}(A_1Q_2)+\frac{N_2A_1A_2}{R_2}+\frac{N_1A_1A_2}{R_1}+q_n^* = 0, \tag{28}$$

where from (19) and (22)

$$\left.\begin{aligned} A_1A_2Q_1 &= \frac{\partial}{\partial\alpha_1}(M_1A_2)+\frac{\partial}{\partial\alpha_2}(M_{21}A_1)+M_{12}\frac{\partial A_1}{\partial\alpha_2}-M_2\frac{\partial A_2}{\partial\alpha_1} \\ A_1A_2Q_2 &= \frac{\partial}{\partial\alpha_1}(M_{12}A_2)+\frac{\partial}{\partial\alpha_2}(M_2A_1)+M_{21}\frac{\partial A_2}{\partial\alpha_1}-M_1\frac{\partial A_1}{\partial\alpha_2} \end{aligned}\right\}. \tag{29}$$

At the boundary

Putting

$$\begin{cases} M_n = \mathbf{n}^T[M]\mathbf{n} \\ N_n = \mathbf{n}^T[N]\mathbf{n} \\ M_{nt} = \mathbf{n}^T[M]\mathbf{t} \\ N_{nt} = \mathbf{n}^T[N]\mathbf{t}, \end{cases}$$

the boundary part of strain energy may be modified as follows:

$$\oint \mathbf{n}^T\Big\{[N]+[M][x]\Big\}\delta\mathbf{D}\,ds+\oint\Big\{-\mathbf{n}^T[M]\delta\,\mathbf{grad}\,w+\mathbf{n}^T\mathbf{Q}\delta w\Big\}ds$$

$$=\oint\left\{\left(N_n+\frac{M_n}{R_n}\right)\delta u_n+\left(N_{nt}+\frac{M_{nt}}{R_t}\right)\delta u_t+Q_n\delta w\right\}ds+\oint\left\{M_n\delta\frac{\partial w}{\partial n}-\frac{\partial}{\partial t}M_{nt}\delta w\frac{\partial}{\partial t}M_{nt}\delta w\right\}ds$$

$$=\oint\left\{\left(N_n+\frac{M_n}{R_n}\right)\delta u_n+\left(N_{nt}+\frac{M_{nt}}{R_t}\right)\delta u_t\right\}ds+\oint\left\{M_n\delta\frac{\partial w}{\partial n}+\left(Q_n+\frac{\partial M_{nt}}{\partial t}\right)\delta w\right\}ds$$

$$+[(M_{nt}(s_i-0)-M_{nt}(s_i+0)]w_i,$$

and the following results are obtained at the boundary for the condition $\delta(W+P)=0$

1. or
$$u_n^* = u_n \qquad N_n^* = N_n \tag{30$_1$}$$

2. or
$$u_t^* = u_t \qquad N_{nt}^* = N_{nt}+\frac{M_{nt}}{R_t} \tag{30$_2$}$$

3. or
$$w^* = w \qquad Q_n^* = Q_n+\frac{\partial M_{nt}}{\partial t} \tag{30$_3$}$$

4. or
$$\frac{\partial w^*}{\partial n} = \frac{\partial w}{\partial n} \qquad M_n^* = M_n \tag{30$_4$}$$

5. or
$$w_i^* = w \qquad Z_i = M_{nt}(s_i-0)-M_{nt}(s_i+0). \tag{30$_5$}$$

As a conclusion, if the continuity of the curvature of the middle surface is secured, a displacement model of Kirchhoff–Love shells is conforming when the following boundary conditions are satisfied at the interface of neighbouring elements:

$$\left.\begin{aligned} (u_{n_+}) &= (u_{n_-}) \\ (u_{t_+}) &= (u_{t_-}) \\ (w)_+ &= (w)_- \\ \left(\frac{\partial w}{\partial n}\right)_+ &= \left(\frac{\partial w}{\partial n}\right)_- \end{aligned}\right\} \tag{31}$$

Notice that the rotation vector is defined as

$$\boldsymbol{\varphi} = -\mathbf{grad}\, w+[\chi]\mathbf{D} = \varphi_1\mathbf{e}_1+\varphi_2\mathbf{e}_2 = \varphi_n\mathbf{n}+\varphi_t\mathbf{t},$$

and

$$\varphi_n = -\frac{\partial w}{\partial n}+\frac{u_n}{R_n}.$$

Finally the conditions (31) may be written as

$$\left.\begin{aligned} (u_{n_+}) &= (u_{n_-}) \\ (u_{t_+}) &= (u_{t_-}) \\ (w)_+ &= (w)_- \\ (\varphi_{n_+}) &= (\varphi_{n_-}) \end{aligned}\right\} \tag{32}$$

4. DUALITY IN THE THEORY OF KIRCHHOFF–LOVE SHELLS

Equations (32) prove that a conforming displacement model requires *a priori* the continuity of the quantities u_n, u_t, w, φ_n.

To develop such an element, one must formulate suitable assumptions about these quantities and define generalized displacement coordinates that determine completely the conformity along each edge. The equilibrium of the element will be insured by the variational principle of total strain energy. In absence of surface forces, the equilibrium equations obtained by the variational principle are, by (27) and (28):

$$\left.\begin{aligned} &\frac{\partial}{\partial \alpha_1}(N_1 A_2)+\frac{\partial}{\partial \alpha_2}(A_1 N_{21})+\frac{\partial A_1}{\partial \alpha_2} N_{12}-N_2 \frac{\partial A_2}{\partial \alpha_1}+\frac{Q_1}{R_1}=0 \\ &\frac{\partial}{\partial \alpha_2}(N_2 A_1)+\frac{\partial}{\partial \alpha_1}(A_2 N_{12})+\frac{\partial A_2}{\partial \alpha_1} N_{21}-N_1 \frac{\partial A_1}{\partial \alpha_2}+\frac{Q_2}{R_2}=0 \\ &\frac{\partial}{\partial \alpha_1}(A_2 Q_1)+\frac{\partial}{\partial \alpha_2}(A_1 Q_2)+\frac{N_2 A_1 A_2}{R_2}+\frac{N_1 A_1 A_2}{R_1}=0, \end{aligned}\right\} \quad (33)$$

where, by (29), the quantities Q_1, Q_2 are such that:

$$\left.\begin{aligned} A_1 A_2 Q_1 &= \frac{\partial M_1 A_2}{\partial \alpha_1}+\frac{\partial M_{21} A_1}{\partial \alpha_2}+M_{12}\frac{\partial A_1}{\partial \alpha_2}-M_2\frac{\partial A_2}{\partial \alpha_1} \\ A_2 A_1 Q_2 &= \frac{\partial M_2 A_1}{\partial \alpha_2}+\frac{\partial M_{12} A_2}{\partial \alpha_1}+M_{21}\frac{\partial A_2}{\partial \alpha_1}-M_1\frac{\partial A_1}{\partial \alpha_2} \end{aligned}\right\}. \quad (34)$$

These equilibrium equations are, moreover, exact, because they are identical to the projection equations (6) and (7) of the general shell theory. The quantities Q_1, Q_2, defined by equation (34) are identified with the transverse shearing forces, because they express the equilibrium of the shell element in the transverse direction.

To develop conforming models, some assumptions are made on the stress components, such that internal equilibrium is satisfied [2] on the one hand, and the continuity of the stress components defined along the edges is secured on the other hand.

The compatibility is insured by the principle of minimum complementary energy. The variation of the complementary energy is

$$\begin{aligned} \delta\phi = \iint [&\varepsilon_1^* \delta N_1+\varepsilon_2^* \delta N_2+\omega_1^* \delta N_{12}+\omega_2^* \delta N_{21}+\kappa_1^* \delta M_1+\kappa_2^* \delta M_2 \\ &+\tau_1^* \delta M_{12}+\tau_2^* \delta M_{21}] A_1 A_2 \,\mathrm{d}\alpha_1 \,\mathrm{d}\alpha_2 \end{aligned} \quad (35)$$

The compatibility equations may be rediscovered by applying the variational principle if one takes into account that the eight stress components N_1, N_2, N_{21}, N_{12}, M_1, M_2, M_{12}, M_{21} are derived from three independent quantities: $U(\alpha_1, \alpha_2)$, $V(\alpha_1, \alpha_2)$, $W(\alpha_1, \alpha_2)$

(representing generalized Airy stress functions) by the following relations:

$$\left.\begin{aligned} N_1 &= K_2^*(U, V, W) \\ N_2 &= K_1^*(U, V, W) \\ N_{12} &= T_2^*(U, V, W) \\ N_{21} &= -T_1^*(U, V, W) \\ M_1 &= E_2^*(U, V, W) \\ M_2 &= -E_1^*(U, V, W) \\ M_{12} &= -\Omega_2^*(U, V, W) \\ M_{21} &= \Omega_1^*(U, V, W) \end{aligned}\right\} \quad (36)$$

The operators E_1^*, E_2^*, Ω_1^*, Ω_2^*, K_1^*, K_2^*, T_1^*, T_2^* act on the stress functions U, V, W in the same way that the eight modified strain components ε_1^*, ε_2^*, w_1^*, w_2^*, κ_1^*, κ_2^*, τ_1^*, τ_2^* act on the displacement components U, V, W. The definitions of strain components are given previously by the general theory [see formulae (1), (2) and (5)].

Conversely, the conjugated asterisk strain components are related to the stress components by the following dual relations:

$$\left.\begin{aligned} \varepsilon_1^* &= -m_2(u, v, w) \\ \varepsilon_2^* &= -m_1(u, v, w) \\ \omega_1^* &= m_{21}(u, v, w) \\ \omega_2^* &= -m_{12}(u, v, w) \\ \kappa_1^* &= n_2(u, v, w) \\ \kappa_2 &= n_1(u, v, w) \\ \tau_1 &= -n_{21}(u, v, w) \\ \tau_2 &= -n_{12}(u, v, w) \end{aligned}\right\} \quad (37)$$

where the eight operators n_1, n_2, n_{12}, n_{21}, m_1, m_2, m_{12}, m_{21} are related to the displacements u, v, w as the eight stress components to the stress functions U, V, W.

The compatibility conditions are formed by substituting quantities (36) and (37) into expression (35) of the complementary energy:

$$\begin{aligned} \delta\phi = \int\!\!\int [&-n_1\delta E_1^* - n_2\delta E_2^* + n_{12}\delta\Omega_1^* + n_{21}\delta\Omega_2^* - m_1\delta K_1^* - m_2\delta K_2^* + m_{21}\delta T_2^* \\ &+ m_{12}\delta T_1^*]A_1A_2\, d\alpha_1\, d\alpha_2 \end{aligned} \quad (38)$$

Then, by comparison it can be seen that this equation is (except for the sign) similar to the variation of strain energy defined by equation (13); rewritten here as

$$\begin{aligned} \delta W = \int\!\!\int [&N_1\delta\varepsilon_1^* + N_2\delta\varepsilon_2^* + N_{12}\delta\omega_1^* + N_{21}\delta\omega_2^* + M_1\delta\kappa_1^* + M_2\delta\kappa_2^* + M_{12}\delta\tau_1^* \\ &+ M_{21}\delta\tau_2^*]A_1A_2\, d\alpha_1\, d\alpha_2. \end{aligned} \quad (39)$$

Attention must be paid to the fact that the variation of (39) is taken on the displacements u, v, w and the variation of (38) is made on the dual quantities U, V, W.

The previous variational calculations could be avoided, simply by comparing the energy equations (38) and (39), and rewriting equations (33), with a few differences of sign

$$\left.\begin{aligned} -\frac{\partial}{\partial\alpha_1}(n_1A_2)+\frac{\partial}{\partial\alpha_2}(A_1n_{21})+\frac{\partial A_1}{\partial\alpha_2}n_{12}+n_2\frac{\partial A_2}{\partial\alpha_1}+\frac{Q_1}{R_1} &= 0\\ -\frac{\partial}{\partial\alpha_2}(n_2A_1)+\frac{\partial}{\partial\alpha_1}(A_2n_{12})+\frac{\partial A_2}{\partial\alpha_1}n_{21}+n_1\frac{\partial A_1}{\partial\alpha_2}+\frac{Q_2}{R_2} &= 0\\ \frac{\partial}{\partial\alpha_1}(A_2Q_1^*)+\frac{\partial}{\partial\alpha_2}(A_1Q_2)-\frac{n_2A_1A_2}{R_2}-\frac{n_1A_1A_2}{R_1} &= 0 \end{aligned}\right\} \tag{40}$$

$$\left.\begin{aligned} A_1A_2Q_1 &= -\frac{\partial m_1A_2}{\partial\alpha_1}+\frac{\partial m_2A_1}{\partial\alpha_2}+m_{12}\frac{\partial A_1}{\partial\alpha_2}+m_2\frac{\partial A_2}{\partial\alpha_1}\\ A_1A_2Q_2 &= -\frac{\partial m_2A_1}{\partial\alpha_2}+\frac{\partial m_1A_2}{\partial\alpha_1}+m_{21}\frac{\partial A_2}{\partial\alpha_1}+m_1\frac{\partial A_1}{\partial\alpha_2} \end{aligned}\right\}. \tag{41}$$

By substituting the values of $n_1, n_2, n_{12}, n_{21}, m_1, m_2, m_{12}, m_{21}$ defined by equations (37), the equations which should be obtained by the variational principle of complementary energy are immediately found as:

$$\left.\begin{aligned} -\frac{\partial}{\partial\alpha_1}(\kappa_2^*A_2)-\frac{\partial}{\partial\alpha_2}(A_1\tau_1^*)+\frac{\partial A_1}{\partial\alpha_2}\tau_2^*+\kappa_1^*\frac{\partial A_2}{\partial\alpha_1}+\frac{Q_1}{R_1} &= 0\\ -\frac{\partial}{\partial\alpha_2}(\kappa_1^*A_1)-\frac{\partial}{\partial\alpha_1}(A_2\tau_2^*)+\frac{\partial A_2}{\partial\alpha_1}\tau_1^*+\kappa_2^*\frac{\partial A_1}{\partial\alpha_2}+\frac{Q_2}{R_2} &= 0\\ \frac{\partial}{\partial\alpha_1}(A_2Q_1)+\frac{\partial}{\partial\alpha_2}(A_1Q_2)-\kappa_1^*\frac{A_1A_2}{R_2}-\frac{\kappa_2^*A_1A_2}{R_2} &= 0 \end{aligned}\right\} \tag{42}$$

$$\left.\begin{aligned} A_1A_2Q_1 &= \frac{\partial\varepsilon_2^*A_2}{\partial\alpha_1}+\frac{\partial\omega_1^*A_1}{\partial\alpha_2}-\omega_2^*\frac{\partial A_1}{\partial\alpha_2}-\varepsilon_1^*\frac{\partial A_2}{\partial\alpha_1}\\ A_2A_1Q_2 &= \frac{\partial\varepsilon_1^*A_1}{\partial\alpha_2}+\frac{\partial\omega_2^*A_2}{\partial\alpha_1}-\omega_1^*\frac{\partial A_2}{\partial\alpha_1}-\varepsilon_2^*\frac{\partial A_1}{\partial\alpha_2} \end{aligned}\right\}. \tag{43}$$

Equations (43) may be transformed as follows:

$$\left.\begin{aligned} Q_1 &= \frac{\phi_1(U,V,W)}{R_2}+\frac{1\partial\Theta}{A_2\partial\alpha_2}(U,V,W)\\ Q_2 &= \frac{\phi_2(U,V,W)}{R_2}-\frac{1\partial\Theta}{A_1\partial\alpha_1}(U,V,W) \end{aligned}\right\} \tag{44}$$

where ϕ_1, ϕ_2, Θ are dual quantities of $\varphi_1, \varphi_2, \theta$ defined by relations (2) and (3).

Replacing now Q_1, Q_2 by their values (43) into equations (42) one obtains

$$\left.\begin{aligned}
&\frac{\partial}{\partial\alpha_1}(A_2\kappa_2^*)+\frac{\partial A_1}{\partial\alpha_2}(\tau_2^*)-\frac{\partial}{\partial\alpha_2}(A_1\tau_1^*)-\frac{\partial A_2}{\partial\alpha_1}\kappa_1^*-\frac{1}{R_1}\left[\frac{1}{\partial\alpha_1}(A_2\varepsilon_2^*)+\frac{\partial A_1}{\partial\alpha_2}\omega_2^*\right.\\
&\qquad\left.-\frac{\partial}{\partial\alpha_2}(A_1\omega_1^*)-\frac{\partial A_2}{\partial\alpha_1}\varepsilon_1^*\right]=0\\
&\frac{\partial}{\partial\alpha_2}(A_1\kappa_1^*)+\frac{\partial A_2}{\partial\alpha_1}\tau_1^*-\frac{\partial}{\partial\alpha_1}(A_2\tau_2^*)-\frac{\partial A_1}{\partial\alpha_2}\kappa_2-\frac{1}{R_2}\left[\frac{\partial}{\partial\alpha_2}(A_1\varepsilon_1^*)+\frac{\partial A_2}{\partial\alpha_1}\omega_1^*\right.\\
&\qquad\left.-\frac{\partial}{\partial\alpha_1}(A_2\omega_2^*)-\frac{\partial A_1}{\partial\alpha_2}\varepsilon_2^*\right]=0\\
&A_1A_2\left(\frac{\kappa_2^*}{R_1}+\frac{\kappa_1^*}{R_2}\right)-\frac{\partial}{\partial\alpha_2}\left\{\frac{1}{A_2}\left[-\frac{\partial}{\partial\alpha_1}(A_2\omega_2^*)+\frac{\partial A_1}{\partial\alpha_2}\varepsilon_2^*-\frac{\partial}{\partial\alpha_2}(A_1\varepsilon_1^*)+\frac{\partial A_2}{\partial\alpha_1}\omega_1^*\right]\right\}\\
&\qquad+\frac{\partial}{\partial\alpha_1}\left\{\frac{1}{A_1}\left[\frac{\partial}{\partial\alpha_1}(A_2\varepsilon_2^*)+\frac{\partial A_1}{\partial\alpha_2}\omega_2^*-\frac{\partial}{\partial\alpha_2}(A_1\omega_1^*)-\frac{\partial A_2}{\partial\alpha_1}\varepsilon_1^*\right]\right\}=0
\end{aligned}\right\} \quad (45)$$

Equations (45) are identical to the compatibility conditions (4) of the general shell theory. *Consequently if the generalized Airy functions* U, V, W *are chosen in such a way that the derived stress components are defined by equations* (36), *the internal compatibility is secured by the variational principle of the complementary energy.*

5. CONTINUITY OF STRESSES AT THE BOUNDARY

If the functions U, V, W are conforming along the edges in the same way as the displacements u, v, w, the stress transmission is perfectly continuous. Indeed, let us now consider an arbitrary face normal to the shell surface element along the α_1 direction of the orthogonal curvilinear coordinate system. In these conditions, the derivation operators become:

$$\left.\begin{aligned}
\frac{\partial}{A_1\partial\alpha_1}&=\frac{\partial}{\partial t}\\
\frac{\partial}{A_2\partial\alpha_2}&=\frac{\partial}{\partial n}
\end{aligned}\right\}. \quad (46)$$

Moreover, the variational calculations given above shows that the Kirchhoff equivalent stress components are four (Fig. 2) and may be related to the five stress components defined on an arbitrary edge by the relations.

$$\left.\begin{aligned}
Q_n^* &= Q_n+\frac{\partial M_{nt}}{\partial t}\\
N_{nt}^* &= N_{nt}+\frac{M_{nt}}{R_t}\\
N_n^* &= N_n\\
M_n^* &= M_n
\end{aligned}\right\}, \quad (47)$$

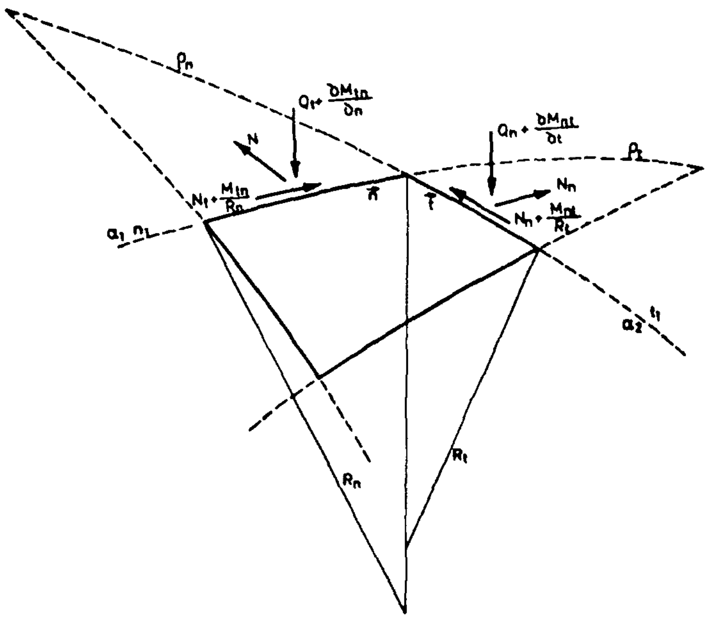

FIG. 2. The equivalent stress components, the geodesic radiuses of curvature and the principal radiuses of curvature.

which, in their turn, may be directly deduced from the quantities U, V, W by analogy (36) and equations (46)

$$\left.\begin{aligned} N_n^* &= \frac{\partial}{\partial t}\left(-\frac{\partial W}{\partial t}+\frac{U_n}{R_t}\right)+\frac{1}{\rho_t}\left(-\frac{\partial W}{\partial n}+\frac{U_t}{R_n}\right) \\ M_n^* &= -\left(\frac{\partial U_n}{\partial t}+\frac{U_t}{\rho_t}+\frac{W}{R_t}\right)\ . \\ N_{nt}^* &= -\frac{\partial \phi_n}{\partial t}+\frac{\phi_t}{\rho_t}+\frac{1}{R_t}\left(\frac{\partial U_t}{\partial t}-\frac{U_n}{\rho_t}\right) \end{aligned}\right\} \tag{48}$$

To obtain Q_n, one must refer to equations (44) which determine the value of Q_n

$$Q_n = \frac{\Phi_n}{R_n}-\frac{\partial \Theta}{\partial t},$$

and then

$$Q_n^* = Q_n+\frac{\partial M_{nt}}{\partial t} = \frac{\Phi_n}{R_n}+\frac{\partial}{\partial t}\left(\frac{U_t}{\partial t}-\frac{U_n}{\rho_t}\right), \tag{49}$$

with

$$\Phi_n = -\frac{\partial W}{\partial n}+\frac{U_t}{R_n}$$

$$\Phi_t = -\frac{\partial W}{\partial t}+\frac{U_n}{R_t}.$$

Equations (48) and (49) prove that the equilibrium at an interface is equivalent to the continuity of the stress functions U, V, W, $\partial W/\partial n$ along this interface. These conditions are proper to conforming models, which require *a priori* the continuity of dual quantities u_n, u_t, w, $\partial w/\partial n$.

The conclusion is that, to each conforming element of shells with double curvature, one may let correspond a perfect equilibrium element, and vice versa. Introducing suitable assumptions on the displacements field for compatible shell elements is equivalent to formulating the generalized Airy functions in equilibrium shell elements.

This duality is a generalization of the results obtained by Fraeijs de Veubeke and Zienkiewicz [3]. In fact, in the case of a plane membrane, we have to consider only the following strain components:

$$\varepsilon_x = \frac{\partial u}{\partial x}, \qquad \varepsilon_y = \frac{\partial v}{\partial y}, \qquad \omega = \frac{\gamma_{xy}}{2} = \frac{1}{2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right),$$

which may be deduced from equations (1). One observes that here:

$$\frac{1}{R_1} = \frac{1}{R_2} = \frac{1}{\rho_1} = \frac{1}{\rho_2} = 0. \tag{50}$$

These strain components possess, by (36), the conjugate quantities

$$M_x = -\frac{\partial V}{\partial y}, \qquad M_y = \frac{\partial U}{\partial x}, \qquad M_{xy} = \frac{1}{2}\left(\frac{\partial V}{\partial x}+\frac{\partial U}{\partial y}\right),$$

which are just the stress components of a plate in bending. The symbols U and V are, except for the sign, the Southwell stress functions.

Therefore, each conforming element of a plane membrane carries a perfect equilibrium element of a bending plate, and vice versa.

Moreover, in the Kirchhoff theory of bent plates, the curvatures generated by the internal bending moments are related to the transverse displacement W by the relations

$$\kappa_x = -\frac{\partial^2 w}{\partial x^2}, \qquad \kappa_y = -\frac{\partial^2 w}{\partial y^2}, \qquad \kappa_{xy} = \kappa_{yx} = -\frac{\partial^2 w}{\partial x \partial y},$$

which may be deduced from equations (2) under the same conditions as (50). According to the duality (36), the conjugate stress components are:

$$N_x = -\frac{\partial^2 W}{\partial y^2}, \qquad N_y = -\frac{\partial^2 W}{\partial x^2}, \qquad N_{xy} = N_{yx} = \frac{\partial^2 W}{\partial x \partial y},$$

which are just the stress components in the plane membrane. The quantity W is except for the sign the usual Airy stress function.

Therefore, to each conforming element of bent plate corresponds a perfect equilibrium element of a plane membrane, and vice versa.

On the other hand, in the membrane shell theory, only the following stress components are considered:

$$N_1, N_2, N_{12}, N_{21}.$$

According to (36) and (37), the determination of these quantities is equivalent to that of the curvature quantities

$$\kappa_2^*, \kappa_1^*, \tau_2^*, \tau_1^*.$$

Therefore there is a connection between the theory of infinitesimal flexure and the membrane theory of shells.

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Resumé—La dualité statico–géométrique est une propriété particulièrement utile dans l'application de la méthode des éléments finis au calcul des coques à double courbure. Après avoir fait ressortir les conditions de bord des coques de Kirchhoff–Love et retrouvé la dualité par le principe variationnel de l'énergie, ce mémoire se propose de dégager le fait qu'à chaque modèle de déplacement pur des coques, on peut faire correspondre un modèle d'équilibre pur et vice-versa. Ceci constitue une généralisation de la dualité existante entre les membranes planes en extension et des plaques en flexion, propriété remarquée il y a quelques années par Fraeijs de Veubeke et Zienkiewicz.

Абстракт—Статико-геометрический дуализм является специально полезным свойством для применения метода конечного злемента к расчету оболочек двойной кривизны. После установления граничных условий для оболочек и определения дуализма, с помощью вариационного принципа, указывается в работе, что каждый соответствующий злемент может согласовываться с елементом равновесия и наоборот. Зто является обобщением стержневой аналогии между изгибом пластинок и растяжением мембран, открытым несколько лет назад фраеийсом де Ведеке и Зенкевичем.